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# ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

ORMOND STONE W. E. BYERLY H. S. WHITE
W. F. OSGOOD F. S. WOODS

PUBLISHED UNDER THE AUSPICES OF HARVARD UNIVERSITY

SECOND SERIES Vol. 3 No. 1

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1901 - 1902

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# ON THE CONVERGENCE OF THE CONTINUED FRACTION OF GAUSS AND OTHER CONTINUED FRACTIONS.*

By Edward B. Van Vleck.

In the *Disquisitiones generales circa seriem infinitam* . . . of Gauss the quotient

$$G(a, \beta, \gamma, x) = \frac{F(a, \beta + 1, \gamma + 1, x)}{F(a, \beta, \gamma, x)}$$

was developed into the continued fraction

$$\frac{1}{1 -} \frac{c_2 x}{1 -} \frac{c_3 x}{1 -} \cdots,$$

in which

$$c_{2n} = \frac{(a + n - 1)(\gamma - \beta + n - 1)}{(\gamma + 2n - 2)(\gamma + 2n - 1)}, \quad c_{2n+1} = \frac{(\beta + n)(\gamma - a + n)}{(\gamma + 2n - 1)(\gamma + 2n)}.$$

The convergence of this continued fraction was later a subject of investigation by Thomé† and Riemann.‡ Both writers showed that, if the portion of the real axis between $x = 1$ and $x = +\infty$ is regarded as a cut, the continued fraction will converge in the remainder of the plane of $x$ except at the points in which $F(a, \beta, \gamma, x)$ and its analytic continuation vanish, and that the limit of the continued fraction is the analytic function defined by $G(a, \beta, \gamma, x)$.§ The proofs of Thomé and Riemann are both of a special and somewhat involved character, not based upon considerations which can be applied to prove the convergence of any considerable class of continued fractions. Indeed, it is only within the past decennium that general theorems for the convergence

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* This paper was read before the American Mathematical Society at its summer meeting in Ithaca, Aug. 20, 1901.

† *Crelle*, vol. 67 (1867), p. 299.

‡ Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita, *Gesammelte Werke*, 1st ed., p. 400. The completion of this posthumous fragment is due to Schwarz.

§ An elementary proof for the special case in which $0 < x < 1$ is found in Schlömilch, *Algebr. Analysis*, p. 321.

of algebraic* continued fractions† have been obtained. The object of this paper is to demonstrate the convergence of the continued fraction of Gauss by theorems which are of the latter character. The theorems employed for this purpose are new and cover a wide class of algebraic continued fractions. Further application of these theorems is made to the quotient of two related Bessel's functions, $J_{n-1}/J_n$, and to the continued fraction which Heine has given as a generalization of that of Gauss.

## I. The Fundamental Theorems.

**1.** *Statement of the First Two Theorems.* To most of the common algebraic continued fractions one or both of the two following theorems may be applied:

Theorem 1. *If $k$ denotes the greatest modulus of a point of condensation of the coefficients $a_n$ of the continued fraction*

$$\frac{a_1}{1+}\ \frac{a_2 x}{1+}\ \frac{a_3 x}{1+}\ \cdots\cdots, \tag{1}$$

*the continued fraction will represent an analytic function within a circle of radius $1/4k$, described about the origin of the $x$-plane as centre, and the only singularities of the function contained within the circle will be poles.*

*If, furthermore, any circle of smaller radius is drawn about the origin and from this circle each of the poles is excluded by drawing around it a small but arbitrary contour, then within the region remaining the continued fraction will converge uniformly to the analytic function as its limit.*

Theorem 2. *If from and after some fixed point in the continued fraction*

$$\frac{1}{b_1 z+}\ \frac{1}{b_2+}\ \frac{1}{b_3 z+}\ \frac{1}{b_4+}\ \cdots\cdots \tag{2}$$

*$b_n$ is real and positive,‡ and $\sum_{n=1}^{\infty} b_n$ is divergent, the continued fraction will converge over the entire plane of $z$ with the exception of*

(1) *the whole or a part of the negative half of the real axis;*

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* Continued fractions whose partial quotients are functions of one or more variables.

† A summary of the few theorems obtained hitherto will be found in the *Transactions of the American Mathematical Society*, vol. 2, 1901, p. 215.

‡ The theorem holds also if from and after some fixed point $b_n$ is negative.

(2) *a series of isolated points* $p_1, p_2, \ldots$, *which lie without this half of the axis.**

*The convergence, moreover, is uniform in any region from which these points and the negative half-axis are excluded after the manner of Theorem* 1, *and the limit of the continued fraction is an analytic function whose only singularities exterior to the negative half-axis are the points* $p_1, p_2, \ldots$, *which are poles.*

**2.** *Proof of these Theorems.* The proof of Theorem 1 will be found in the *Transactions of the American Mathematical Society*, vol. 2, October, 1901. Theorem 2 may be demonstrated as follows.

By hypothesis, there is some point in (2) subsequent to which $b_n$ is real and positive. Suppose this to be true when $n \geqq 2\mu$, and consider the fraction

$$\frac{1}{b_{2\mu+1}z} + \frac{1}{b_{2\mu+2}} + \frac{1}{b_{2\mu+3}z} + \cdots \tag{3}$$

If the $n$th convergent of (2) is denoted by $N_n/D_n$ and the $m$th convergent of (3) by $N'_m/D'_m$, we obviously have

$$\frac{N_{2\mu+m}}{D_{2\mu+m}} = \frac{N_{2\mu} + \dfrac{N'_m}{D'_m} N_{2\mu-1}}{D_{2\mu} + \dfrac{N'_m}{D'_m} D_{2\mu-1}}. \tag{4}$$

Hence to ascertain whether the successive convergents of (2) approach a limit, it is sufficient first to determine whether this is the case with the convergents of (3).

Now Stieltjes† has proved that when the coefficients $b_{2\mu+n}$ in the continued fraction (3) are positive for all values of $n$ and $\Sigma b_{2\mu+n}$ is divergent, the continued fraction will converge uniformly in any region $T$, bounded by a closed curve which has no point in common with the negative half of the axis of $z$, and the limit of the fraction is a function which is holomorphic in $T$. It follows that when $m$ is indefinitely increased, $N'_m/D'_m$ converges uniformly within any such region, and its limit $\phi(z)$ is holomorphic at every

* It is proved later (§3) that if $b_n$ fulfills the conditions of the theorem and is also real from the beginning of the continued fraction, the number of the points $p_n$ must be finite. Whether this number is always finite does not appear; but if the number is infinite, the derivative of the point-set obviously can consist only of points which lie upon the negative half of the axis.

† *Annales de Toulouse*, vol. 8. Another proof is given by the writer in the *Transactions Amer. Math. Society*, vol. 2, 1901; see in particular p. 231–2.

point without the negative half-axis. Since the numerator and denominator of the right hand member of (4) also converge in the same manner, their quotient must converge uniformly to the limit

$$\frac{N_{2\mu} + \phi(z)\ N_{2\mu-1}}{D_{2\mu} + \phi(z)\ D_{2\mu-1}} \tag{5}$$

except in the vicinity of the roots of $D_{2\mu} + \phi(z)\ D_{2\mu-1}$ or along the negative half of the axis. It is impossible for $D_{2\mu} + \phi(z)\ D_{2\mu-1}$ to vanish identically, since in that case the numerator would have to vanish identically too, and hence $N_{2\mu} D_{2\mu-1} - N_{2\mu-1} D_{2\mu} \equiv 0$, which is not true. As the numerator and denominator of (5) are also holomorphic at all points without the half-axis, the expression (5) is an analytic function whose only singularities exterior to the half-axis are poles. The continued fraction (2) therefore converges in the manner stated in the theorem.*

**3.** *A Sufficient Condition that the Number of the Poles be Finite.* In at least one case (which is stated below in Theorem 3) we can easily prove that the number of poles $p_1, p_2, \ldots$ is finite.

When, namely, the coefficients in the continued fraction (2) are real for all values of $n$ and satisfy the conditions of Theorem 2, an upper limit can be found for the number of poles of the limiting function exterior to the negative half of the axis. These poles, in fact, correspond to the zeros of the analytic function to which $D_n$ converges as $n$ increases indefinitely, and the convergence is obviously uniform in the vicinity of each zero which does not lie upon the negative half-axis. Now it is well known that when a series of functions $f_n(z)$, holomorphic throughout any region $T$, converges uniformly in this region, the zeros of the limiting function within $T$ are the points of condensation of the zeros of the functions $f_n(z)$. Here we have $f_n(z) = D_{2\mu} + \frac{N_n'}{D_n'} D_{2\mu-1}$. Furthermore, Hurwitz† has shown that if a circle

* For the sake of completeness it may be added that when $\Sigma b_n$ is convergent, the even convergents of (2) and the odd convergents approach separate limits, so that it is impossible to speak of the convergence of the continued fraction or of a single limiting function which it represents. This is always the case when $\Sigma b_n$ converges, irrespective of other conditions which may be imposed on $b_n$, and the limits of the two series of convergents are meromorphic over the entire plane. This result is not new. See, for example, the memoir of Stieltjes previously cited. For another mode of proof see the *Transactions Amer. Math. Soc.*, vol. 2, 1901, p. 231.

† *Math. Annalen*, vol. 33, 1888, p. 249.

of sufficiently small radius is drawn about each zero of the limiting function within $T$, each of the circles will contain a number of roots of $f_n(z)$ exactly equal to the order of the zero enclosed, for all values of $n \geqq m$, where $m$ is a fixed integer. Our question concerning the number of poles is thus reduced to a determination of an upper limit for the number of roots of $D_n$ exterior to the negative half of the axis.

Let $\nu$ be the smallest value of $n$ subsequent to which $b_n$ is always positive, and consider then the number of changes of sign in the series

$$D_\nu, \quad D_{\nu+2}, \quad D_{\nu+4}, \quad \ldots, \quad D_{\nu+2n} \tag{6}$$

for any real value of $z$. When $z$ varies continuously, this number will change only when one or more of the $D$'s vanish. From the fundamental relation connecting the denominators of three successive convergents it will be found that

$$D_n = \left( b_n b_{n-1} z + 1 + \frac{b_n}{b_{n-2}} \right) D_{n-2} - \frac{b_n}{b_{n-2}} D_{n-4}, \tag{7}$$

alike when $n$ is odd or even. Accordingly, when any term in (6) vanishes, the preceding and following terms have opposite signs. If two successive terms should vanish, then simultaneously every $D_{\nu \pm 2i}$ vanishes. But $D_0 \equiv 1$, $D_1 \equiv b_{2n+1} z$. This happens therefore only when $\nu$ is odd and $z = 0$. Hence we see that an alteration in the number of changes of sign in (6) can take place only when $z$ passes through a root of $D_\nu$ or $D_{\nu+2n}$, and that there is a loss or a gain of a single sign for a root of either unless $z = 0$.

Suppose now that $\nu$ is even. If we throw out a common factor, the terms of highest degree in the successive members of (6) will be equal to

$$1, \quad b_{\nu+1} b_{\nu+2} z, \quad b_{\nu+1} b_{\nu+2} b_{\nu+3} b_{\nu+4} z^2, \quad \ldots\ldots, \quad b_{\nu+1} \ldots b_{\nu+2n} z^n. \tag{8}$$

When, therefore, $z = -\infty$ is substituted in (6), there will be $n$ alterations of sign. On the other hand, the terms of (6) all have the same sign when $z = 0$, since the constant term of $D_{2n}$ is always equal to 1. It follows that $D_\nu$ and $D_{\nu+2n}$ together must have at least $n$ roots in the negative half-axis. But their degrees are equal respectively to $\nu/2$ and $n + \nu/2$. We conclude therefore that the number of roots of $D_{\nu+2n}$ without the half-axis can not exceed $\nu$.

Suppose next that $\nu$ is odd. As before, there will be $n$ changes of sign in (6) when $z = -\infty$. The terms of lowest degree, however, are equal now to

$$(b_1 + b_3 + b_5 + \cdots + b_\nu)z, \quad (b_1 + b_3 + \cdots + b_{\nu+2})z, \cdots,$$
$$(b_1 + b_3 + \cdots + b_{\nu+2n})z.$$

Since $b_n$, by hypothesis, is positive for $n > \nu$, the number of changes of sign here can not exceed unity. Hence the number of changes of sign in (6) for an exceedingly small negative value of $z$ cannot exceed unity, and $D_\nu$ and $D_{\nu+2n}$ together have at least $n - 1$ roots between $z = 0$ and $z = -\infty$. As they also each have a root in the origin and as their degrees are equal respectively to $(\nu + 1)/2$ and $n + (\nu + 1)/2$, we conclude again that the number of roots of $D_n$ exterior to the negative half of the axis will not exceed $\nu$. We reach thus the following result:

THEOREM 3. *If the coefficients $b_n$ are real from the outset in the continued fraction of Theorem 2 and if $\nu$ is the smallest value of $n$ subsequent to which they are always positive, the number of poles of the function defined by the continued fraction, exterior to the negative half of the real axis of $z$, will not exceed $\nu$, each multiple pole being counted a number of times equal to its order.*

**4.** *A Sufficient Condition for the Applicability of Theorem* 1. We proceed now to determine a class of continued fractions of the form (1) to which Theorem 1 is applicable. In many continued fractions of that form, it is possible, for sufficiently large values of $n$, to expand $a_{2n-1}/a_{2n}$ and $a_{2n}/a_{2n+1}$ into convergent series in ascending powers of $1/n$ with constant coefficients. Suppose then that

$$\frac{a_{2n-1}}{a_{2n}} = 1 + \frac{A_1'}{n} + \frac{A_2'}{n^2} + \cdots, \tag{9}$$

$$\frac{a_{2n}}{a_{2n+1}} = 1 + \frac{A_1''}{n} + \frac{A_2''}{n^2} + \cdots \tag{10}$$

From these we derive

$$\frac{a_{2n+1}}{a_{2n-1}} = 1 - \frac{A_1' + A_1''}{n} + \frac{1}{n^2}(\ ) + \cdots, \tag{11}$$

and a similar power series for $a_{2n+2}/a_{2n}$, in which the first two terms are the same as in (11). If now the real part of $A_1' + A_1''$ is positive, a theorem of Weierstrass* shows that $a_{2n+1}$ and $a_{2n}$ with increasing $n$ approach the limit 0.

* Ueber die Theorie der analytischen Facultäten, *Crelle*, vol. 51, 1855, p. 25; *Werke*, vol. 1, p. 181.

Theorem 1 then gives complete information concerning the continued fraction and shows that it represents a function which is either holomorphic or meromorphic over the entire finite plane.

If the real part of $A_1' + A_1''$ is negative, $a_n$ approaches the limit $k = \infty$, and no information at all concerning the continued fraction can be obtained from Theorem 1.

Lastly, if the real part of $A_1' + A_1''$ is equal to zero, the moduli of $a_{2n}$ and $a_{2n+1}$ approach finite limits, which by (10) are seen to be identical. Their arguments, however, will fail to approach definite values, unless the imaginary part of $A_1' + A_1''$ vanishes, too. When this happens, $a_{2n}$ and $a_{2n+1}$ must both approach limiting values, since $\prod_{n=1}^{\infty}(1 + \frac{1}{n^2})$ is convergent. The identity of the two limits then follows from (10). *If, therefore, equations* (9) *and* (10) *hold for large values of* $n$, *the necessary and sufficient condition that* $a_n$ *shall approach a finite limit different from zero is that* $A_1' + A_1''$ *shall be equal to zero.*

**5.** *The Convergence of the Continued Fraction* (1) *when* $A_1' + A_1'' = 0$. *Case* I: $a_n$ *Ultimately Real.* When the condition:

$$A_1' + A_1'' = 0$$

is fulfilled, Theorem 1 determines the convergence of the continued fraction for a portion of the $x$-plane.* Further information regarding its convergence can be gained by throwing it into the form (2). For this purpose put $x = 1/z$ and then transform so as to remove $z$ from the partial numerators. It will then appear in the alternate denominators, beginning with the first. An easy and well known reduction, which alters neither the value of the continued fraction nor the value of its convergents, then gives the following values for the coefficients of (2):

$$b_1 = \frac{1}{a_1}, \quad b_{2n} = \frac{a_1 a_3 \cdot \cdot \cdot a_{2n-1}}{a_2 a_4 \cdot \cdot \cdot a_{2n}}, \quad b_{2n+1} = \frac{a_2 a_4 \cdot \cdot \cdot a_{2n}}{a_1 a_3 \cdot \cdot \cdot a_{2n+1}}. \qquad (12)$$

We will take up first the case in which $a_n$ *is real for all values of* $n \geqq m$, where $m$ denotes a fixed number. Since by hypothesis

$$\lim_{n=\infty} \frac{a_{n-1}}{a_n} = 1,$$

* Theorem 1 is also applicable when only the real part $A_1' + A_1''$ vanishes. This case will not concern us here.

$a_n$ must ultimately have a constant sign. If this sign is $-$, it may be changed to $+$ by putting $x = -x'$. We may therefore assume that $a_n$ is real and positive when $n$ exceeds the even integer $2\mu$.

Let now the first $2\mu$ partial quotients of (1) be omitted, and consider the continued fraction which remains. For convenience we will denote the coefficients of its numerators by $a'_1, a'_2, \cdots$. The corresponding values of $b'_{2n}$ and $b'_{2n+1}$ are real and positive. Furthermore, when $n$ is sufficiently increased, we have

$$\frac{b'_{2n}}{b'_{2n-2}} = \frac{a'_{2n-1}}{a'_{2n}} = 1 + \frac{A'_1}{n+\mu} + \frac{A'_2}{(n+\mu)^2} + \cdots, \tag{13}$$

$$\frac{b'_{2n+1}}{b'_{2n-1}} = \frac{a'_{2n}}{a'_{2n+1}} = 1 + \frac{A''_1}{n+\mu} + \cdots. \tag{14}$$

Now since $A'_1 + A''_1 = 0$, either $A'_1$ or $A''_1$ must be greater than $-1$: say the former. Then

$$\lim_{n=\infty} n\left(\frac{b'_{2n}}{b'_{2n-2}} - 1\right) > -1,$$

whence it follows by a theorem of Raabe's* that $\sum_{n=1}^{\infty} b'_{2n}$ is divergent. If $A''_1 > -1$, we see similarly that $\Sigma\, b'_{2n+1}$ diverges. In either case, all the conditions of the theorem of Stieltjes quoted in §2 are fulfilled, and the continued fraction, after the omission of the first $2\mu$ partial quotients, will converge over the entire plane of $z$ except along a cut from $z = 0$ to $z = -\infty$. Since $x = 1/z$, there is a similar region of convergence in the plane of $x$. When the $2\mu$ partial quotients are restored, the reasoning of §2 may be repeated. We conclude therefore that under the conditions indicated the continued fraction (1) represents a function whose only singularities exterior to the negative half-axis of $x$ are poles.†

**6.** *Continuation; Case* II: $a_n$ *Not Ultimately Real.* The direct consideration of (1) when $A'_1 + A''_1 = 0$, appears to be difficult if $a_n$ is not

* *Zeitschrift für Physik und Mathematik*, vol. 10, Vienna, 1832, p. 63; *cf.* for example, Stolz, *Allgemeine Arithmetik*, vol. 1, p. 263.

† If $A'_1 < -1$ and $A''_1 < -1$, the series $\Sigma b'_n$ is convergent. In this case the limit of the series of odd convergents or of the series of even convergents is meromorphic over the entire plane of $x$, but the two limits are not identical.

ultimately real, or at least if $A_1'$ and $A_1''$ are not real. For Weierstrass* shows that the argument of $b_n$ will fail to approach a limiting value if the imaginary component of $A_1'$ in (13) or of $A_1''$ in (14) is not equal to zero. On this account it is better to proceed indirectly.

Equation (7) enables us to consider the even and the odd convergents separately. Since a similar relation connects the numerators of the convergents of (2), the even convergents form by themselves the successive convergents of the continued fraction

$$\cfrac{b_2}{b_1b_2z+1-\cfrac{\cfrac{b_4}{b_2}}{b_3b_4z+1+\cfrac{b_4}{b_2}-\cfrac{\cfrac{b_6}{b_4}}{b_5b_6z+1+\cfrac{b_6}{b_4}-}}}\quad\cdots, \tag{15}$$

while the odd convergents have a like connection with

$$\frac{1}{b_1z}\left(1-\cfrac{\cfrac{b_3}{b_1}}{b_2b_3z+1+\cfrac{b_3}{b_1}-\cfrac{\cfrac{b_5}{b_3}}{b_4b_5z+1+\cfrac{b_5}{b_3}}\cdots\cdot}\right). \tag{16}$$

If in (16) we set $\dfrac{b_{2n+1}}{b_{2n-1}}=a_n''$ and then reduce the continued fraction to the normal form in which each partial numerator is equal to 1, it becomes

$$\frac{1}{b_1z}\left(1-\cfrac{1}{b_1''\left(b_2b_3z+1+\dfrac{b_3}{b_1}\right)-\cfrac{1}{b_2''\left(b_4b_5z+1+\dfrac{b_5}{b_3}\right)-}}\cdots\cdot\right), \tag{17}$$

in which $b_n''$ is obtained from the $a_n''$ by doubly accenting the letters of (12).

The continued fraction (17) is of the form

$$1-\cfrac{1}{B_1z+C_1-\cfrac{1}{B_2z+C_2-}}\cdots\cdot \tag{17a}$$

We shall show that $B_n$ approaches a limit different from 0, which we shall denote by $k^{-1}$, and that $C_n$ approaches 2 as its limit, when $n=\infty$. The convergence of the continued fraction

$$1-\cfrac{1}{k^{-1}z+2-\cfrac{1}{k^{-1}z+2-}}\cdots\cdot \tag{17b}$$

* *Werke*, *l. c.*, p. 181–184.

can be dealt with directly by means of theorems already established, and then the fraction (17$a$) can be treated in a similar manner.

Let the first $2r-1$ partial quotients of (1) be omitted and consider in its place the continued fraction which remains. If $r$ has been taken sufficiently large we shall have in the continued fraction corresponding to (17)

$$b_1'' = \frac{b_1}{b_3} = \frac{a_{2r+2}}{a_{2r+1}} = 1 - \frac{A_1''}{r+1} + \frac{(\ )}{(r+1)^2} + \cdots,$$

and the value of $b_1''$ can accordingly be made to differ from unity by as little as we choose. The same is obviously true of $b_2''$. Furthermore

$$\frac{b_{2n}''}{b_{2n-2}''} = \frac{a_{2n}''-1}{a_{2n}''} = \frac{b_{4n-1}}{b_{4n-3}} \div \frac{b_{4n+1}}{b_{4n-1}} = \frac{a_{4n+2r-3}}{a_{4n+2r-2}} \div \frac{a_{4n+2r-1}}{a_{4n+2r}},$$

whence we obtain with the aid of (9) and (10)

$$\frac{b_{2n}''}{b_{2n-2}''} = \frac{1 + \dfrac{A_1'}{2n+r-1} + \dfrac{A_2'}{(2n+r-1)^2} + \cdots}{1 + \dfrac{A_1'}{2n+r} + \dfrac{A_2'}{(2n+r)^2} + \cdots}$$

$$= 1 + \frac{B_2}{(2n+r)^2} + \frac{B_3}{(2n+r)^3} + \cdots.$$

It will be noticed that the first power of $1/(2n+r)$ does not appear. We can show in like manner that it is absent in the expansion of $b_{2n+1}''/b_{2n-1}''$. There must therefore exist a positive number $C$ such that the inequality

$$\left| \frac{b_n''}{b_{n-2}''} \right| < 1 + \frac{C}{(n+r)^2}$$

shall hold for all values of $n$. Now the product

$$\prod_{n=1}^{\infty} \left(1 + \frac{C}{(n+r)^2}\right)$$

by increasing $r$ sufficiently can be made to differ from unity by as little as we please. Since this is also the case with $b_1''$ and $b_2''$, it is possible so to choose $r$ as to make

$$|b_n'' - 1| < \epsilon_1, \qquad (n = 1, 2, \ldots),$$

in which $\epsilon_1$ is a small but arbitrarily assigned positive quantity.

Furthermore, in each partial denominator of (17) we have the relation $b_n b_{n+1} = 1/a_{n+2r}$. If, then, $k$ is the limit of $a_n$, the following condition may also be fulfilled

$$|b_n b_{n+1} - k^{-1}| < \epsilon_1, \qquad (n = 1, 2, \ldots)$$

Lastly, since

$$\frac{b_{n+2}}{b_n} = \frac{a_{n+2r}}{a_{n+2r+1}},$$

the value of this ratio can be made to differ from unity by a quantity of as small a modulus as is desired. Hence we see, finally, that *if $r$ has been taken sufficiently large, the successive partial denominators of* (17) *for any given finite value of $z$ will all differ from $k^{-1}z + 2$ by a quantity whose modulus is smaller than an arbitrarily prescribed quantity $\epsilon$. It is also possible so to choose $r$ that this shall hold uniformly for a given finite region of the plane of $z$.*

**7.** *Completion of the Proof in Case* II. We now have all of the material at hand for the final determination of the region of convergence of the continued fraction (1). We suppose that $r$ fulfills the condition of the close of §6 and place

$$w = k^{-1}z.$$

Also let $T'$ be the corresponding region in the plane of $w$. This region shall lie wholly above or else wholly below the real axis. If the continued fraction (17) is transformed so as to make each of its numerators equal to + 1, the signs of the alternate denominators must be changed. Hence at each point of $T'$ the imaginary component of the denominators will be alternately positive and negative. Also, in $T'$ the absolute value of the ratio of the imaginary to the real component of any denominator will never sink below a finite minimum. Lastly, the sum of the absolute values of the denominators is divergent throughout $T'$. When these three conditions are fulfilled, it follows by a theorem which I have given in the *Transactions of the American Mathematical Society*, vol. 2, p. 229, that the continued fraction converges throughout $T'$. The reasoning of the succeeding pages also shows that the convergence is uniform throughout $T'$.

We have thus shown, under the conditions stated, that if a sufficient number of partial quotients at the outset in (1) are omitted, the convergents of odd order will converge at each point of $T'$. A similar course of reasoning applied to (15) will prove that this is also true of the even convergents. Since

the convergence is uniform, the limits are holomorphic within $T''$. If, now, the omitted partial quotients are restored, the reasoning of §2 makes it evident that the limits of the two series of convergents are analytic functions whose only singularities in the interior of $T''$ are poles. Since $w = k^{-1}x^{-1}$, it follows from Theorem 1 that the limits of the two series of convergents are identical in the portion of the $w$-plane which lies without a circle described about the origin as center with a radius equal to 4. If $T''$ does not extend into this portion of the plane, it is possible to enlarge it so that it shall. Thus the limits of the odd and of the even convergents must coincide throughout the whole region $T''$ since they are analytic functions and coincide throughout a part of this region. Finally $T''$ can be enlarged so as to include as much of the positive (or of the negative) half-plane as is desired. We conclude therefore that if a cut is made along the real axis from $w = -4$ to $w = 4$, the continued fraction will represent a function which is meromorphic in the remainder of the plane.

To obtain the corresponding region of the $x$-plane, we first draw a straight line through the points $x = \pm 1/4k$ and then cut the plane along the two segments of the line which lie between these points and $\infty$.

**8.** Our last step will be to show that one of the two cuts just made may be dispensed with. Take for this purpose any region $T'''$ in the $w$-plane in which the real part of $w$ is positive and which includes a portion of the real axis. We have already proved that by properly choosing $r$ the denominators in the continued fraction (17) can be made to differ from $w + 2$ throughout $T''$ by a quantity whose modulus is smaller than an arbitrarily prescribed quantity $\epsilon$. If $\epsilon$ is taken sufficiently small, the moduli of the denominators will exceed 2 at each point of $T'''$, inasmuch as the real part of $w$ is positive. This will also hold for each denominator, except the first, in the continued fraction which we derive from (15) in the same manner as (17) was derived from (16). The first denominator, $b_2^{-1}(b_1 b_2 k w + 1)$, will obviously exceed 1 by at least some small quantity $h$, if $r$ is sufficiently large. The convergence of both continued fractions within $T'''$ then follows from a theorem of Pringsheim.* We shall, however, reproduce his reasoning here in such form as to make it immediately apparent that the convergence of both fractions is uniform.

For convenience, let the $n$th partial quotient in either continued fraction

* *Sitzungsberichte der Münchener Akademie*, vol. 28, 1898, p. 313–316.

be denoted by $1/r_n e^{i\theta_n}$. According to our hypothesis we have for each point of $T''$

$$r_1 > 1 + h,$$

$$r_n \geqq 2 \qquad (n > 1),$$

and hence

$$|D_2| - |D_1| \geqq h.$$

From the relation

$$D_n = r_n e^{i\theta_n} D_{n-1} + D_{n-2}$$

it follows that

$$|D_n| - |D_{n-1}| \geqq |D_{n-1}| - |D_{n-2}| \geqq \cdots \geqq h.$$

Hence $|D_n| \geqq 1 + nh$. Furthermore

$$\frac{N_{n+1}}{D_{n+1}} - \frac{N_n}{D_n} = -\frac{1}{D_n D_{n+1}}$$

and consequently

$$\left|\frac{N_{n+i}}{D_{n+i}} - \frac{N_n}{D_n}\right| \leqq \frac{1}{|D_n D_{n+1}|} + \frac{1}{|D_{n+1} D_{n+2}|} + \cdots + \frac{1}{|D_{n+i-1} D_{n+i}|}$$

$$\leqq \frac{1}{h^2}\left[\frac{1}{(n + h^{-1})(n + 1 + h^{-1})} + \frac{1}{(n + 1 + h^{-1})(n + 2 + h^{-1})} + \cdots + \frac{1}{(n + i - 1 + h^{-1})(n + i + h^{-1})}\right].$$

Since the value of the last expression can be made as small as we choose by sufficiently increasing $n$, the continued fraction will converge uniformly within $T''$.

The region $T''$ can now be enlarged so as to include as much of the positive half of the $w$-axis as is desired. The portion of the $w$-cut between the origin and $w = 4$ was therefore superfluous. In the $x$-plane we may omit the cut between $x = 1/4k$ and $x = \infty$ which corresponded to this portion.

This completes our determination of the character of the function represented by (1) when $A_1' + A_1'' = 0$. The result reached can be summed up as follows:

THEOREM 4. *If in the continued fraction of Theorem* 1 *the ratios* $a_{2n-1}/a_{2n}$ *and* $a_{2n}/a_{2n+1}$ *for sufficiently large values of* $n$ *can be expanded into convergent series in powers of* $1/n$ *and if* $A_1' + A_1'' = 0$, *the region within which the continued fraction represents an analytic function can be extended so as to include the entire plane of* $x$ *with the exception of a cut drawn from* $x = -1/4k$ *to* $x = \infty$

*in a direction which forms the continuation of the line joining the origin to* $x = -1/4k$.

*The only singularities of the analytic function exterior to the cut are poles and the continued fraction converges in the same manner as in Theorem* 1.

*If* $A_1' + A_1'' \neq 0$ *and the real part of* $A_1' + A_1''$ *is positive, the continued fraction represents an analytic function which, in the finite region of the plane, has no other singularities than poles.*

*If the real part of* $A_1' + A_1''$ *vanishes but the imaginary part does not, then the continued fraction represents an analytic function which at least within a circle described with a radius* $1/4k$ *about the point* $x = 0$ *has no other singularities than poles.*

## II. Application of the Fundamental Theorems.

**9.** $(a)$. *The Continued Fraction of Gauss.* The last theorem can be applied directly to the continued fraction of Gauss. For since

$$a_{2n} = -\frac{(\alpha - 1 + n)(\gamma - 1 - \beta + n)}{(\gamma + 2n - 2)(\gamma + 2n - 1)}, \qquad a_{2n+1} = -\frac{(\beta + n)(\gamma - \alpha + n)}{(\gamma + 2n - 1)(\gamma + 2n)},$$

$a_n$ has the limit $k = -1/4$. Moreover,

$$\frac{a_{2n-1}}{a_{2n}} = 1 + \frac{2\beta - 2\alpha + 1}{n} + \frac{1}{n^2}(\quad) + \cdots,$$

$$\frac{a_{2n}}{a_{2n+1}} = 1 - \frac{2\beta - 2\alpha + 1}{n} + \cdots,$$

and $A_1' + A_1'' = 0$. The continued fraction therefore represents a function whose only singularities exterior to the portion of the real axis between $x = 1$ and $x = +\infty$ are poles. It also converges uniformly to the function as its limit in any region bounded by a continuous curve, from which the poles and this portion of the axis are excluded.

It remains to identify this function with the quotient $G(\alpha, \beta, \gamma, x)$ from which Gauss derived the continued fraction. Now

$$G(\alpha, \beta, \gamma, x) = \frac{1}{1 + \dfrac{a_2 x}{1 + \dfrac{a_3 x}{1 + \cdots + \dfrac{a_{2n} x}{1 + a_{2n+1} x\, G_1}}}},$$

where

$$G_1 = G(\alpha + n, \beta + n, \gamma + 2n, x).$$

Hence

$$G(a, \beta, \gamma, x) - \frac{N_{2n}}{D_{2n}} = \frac{N_{2n} + a_{2n+1}\, x\, G_1\, N_{2n-1}}{D_{2n} + a_{2n+1}\, x\, G_1\, D_{2n-1}} - \frac{N_{2n}}{D_{2n}}$$

$$= \frac{a_2 a_3 \cdot \cdot \cdot a_{2n+1}\, G_1\, x^{2n}}{D_{2n}(D_{2n} + a_{2n+1}\, x\, G_1\, D_{2n-1})}.$$

This shows that when $G(a, \beta, \gamma, x)$ and $N_{2n}/D_{2n}$ are expanded into series in ascending powers of $x$, their expansions agree to a greater and greater number of terms as $n$ increases. We will prove that $G(a, \beta, \gamma, x)$ coincides with the limit of $N_{2n}/D_{2n}$ by demonstrating the following

THEOREM. *If a sequence of analytic functions converges uniformly to a limit in the vicinity of the origin and their expansions in ascending powers of $x$, as $n$ increases, agree to a greater and greater number of terms with a given series $P(x)$, this series is the limit of the sequence.**

Let $f_n(x)$ represent the $n$th of these functions and let

$$f_n(x) = a_0^{(n)} + a_1^{(n)} x + \cdot \cdot \cdot \cdot,$$

$$P(x) = c_0 + c_1 x + \cdot \cdot \cdot \cdot.$$

Then, by hypothesis, if the positive integer $m$ is chosen at pleasure, a second integer $\mu$ can be so determined that

$$a_0^{(n)} = c_0, \qquad a_1^{(n)} = c_1, \quad \cdot \cdot \cdot \quad a_{m-1}^{(n)} = c_{m-1}, \qquad n \geqq \mu.$$

Hence, generally,

$$\lim_{n=\infty} a_i^{(n)} = c_i.$$

The theorem asserts that

$$P(x) = \lim_{n=\infty} f_n(x).$$

To prove it, then, we have only to show that the limit of the series

$$a_0^{(n)} + a_1^{(n)} x + \cdot \cdot \cdot \cdot$$

---

* This theorem follows from the theorem that, if $s_n(x)$ is a single valued, analytic function of $x$ throughout the two-dimensional region $T$ of the complex plane for $n = 1, 2, \ldots$ and if $s_n(x)$ converges uniformly in $T$ when $n = \infty$, its limit being denoted by $F(x)$; then $F^{(i)}(x) = \lim_{n=\infty} s_n^{(i)}(x)$, $i = 1, 2, 3, \ldots$, and the proof is now readily completed. *Cf.* de la Vallée Poussin, *Annales de la Société scientifique de Bruxelles*, seconde partie, Mémoires, vol. 17, 1893, p. 324. [ED.]

is equal to the series of the limits:

$$\lim_{n=\infty} a_0^{(n)} + \lim_{n=\infty} a_1^{(n)} x + \cdots .$$

This follows immediately from the uniform convergence. Thus the theorem is proven, and the limit of the continued fraction of Gauss is the function $G(\alpha, \beta, \gamma, x)$.

If $\alpha, \beta, \gamma$ are real, an upper limit for the number of poles of this function exterior to the positive half of the real axis can be obtained with the aid of Theorem 3. It will be seen at once that it can not be greater than $2N$, if $N$ is the smallest positive integral value of $n$ for which the following inequalities simultaneously hold:

$$\alpha + n - 1 > 0, \qquad \beta + n > 0$$

$$\gamma - 1 - \beta + n > 0, \qquad \gamma - \alpha + n > 0, \qquad \gamma + 2n > 0.$$

This furnishes also an upper limit for the number of zeros of $F(\alpha, \beta, \gamma, x)$ and its analytic continuation which lie without the positive half-axis.

**10.** $(b)$. *The Continued Fraction for Bessel's Quotient* $J_n/J_{n-1}$. From the well known equation connecting three consecutive related Bessel's functions,

$$\frac{J_{n-1}(x)}{J_n(x)} = \frac{2n}{x} - \frac{J_{n+1}(x)}{J_n(x)},$$

the continued fraction

$$\frac{2n}{x} - \cfrac{1}{\frac{2n+2}{x} - \cfrac{1}{\frac{2n+4}{x} - \cdots}}$$

is derived. This may be transformed so as to become

$$\frac{2n}{x} - \cfrac{\frac{x}{2n+2}}{1 - \cfrac{\frac{x^2}{(2n+2)(2n+4)}}{1 - \cfrac{\frac{x^2}{(2n+4)(2n+6)}}{1 - \cdots}}} \tag{18}$$

Also by removing a factor $x^{-1}$ and placing $z = -x^{-2}$, it may be thrown into the form

$$2n + \cfrac{1}{(2n+2)z + \cfrac{1}{(2n+4) + \cfrac{1}{(2n+6)z + \cdots}}} \tag{19}$$

The application of Theorem 1 to the continued fraction (18) shows that it represents a function which is meromorphic over the entire plane. The

identity of the function with $J_{n-1}/J_n$ can be established easily with the aid of the theorem of §9.

When $n$ is real, Theorems 2 and 3 can both be applied to (19). We thus find that the number of poles of the function defined by (19) which lie without the negative half of the axis is equal to zero if $n > -1$. When $n$ lies between the two negative integers, $-m$ and $-(m+1)$, the integer $m$ furnishes an upper limit for the number of such poles. Now to each of these poles there corresponds a pair of imaginary poles of the meromorphic function defined by (18). Since the poles of the latter function are the roots of $J_n$, we conclude that all the roots of $J_n$ are real if $n > -1$. If $n < -1$, the number of pairs of conjugate imaginary roots can not exceed $m$. As a matter of fact, this is the exact number.*

**11.** (*c*). *Heine's Continued Fraction.* One of the generalizations of the hypergeometric series $F(\alpha, \beta, \gamma, x)$, which was proposed by Heine† is the series

$$\phi(\alpha, \beta, \gamma, q, x) = 1 + \frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})} + \frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})} + \cdots.$$

This satisfies a difference equation of the second order. The hypergeometric series is the limiting case obtained by letting $q$ approach unity. Without in any way restricting the generality of the above series, we may assume $q$ to be less than unity.

For the quotient

$$\frac{\phi(\alpha, \beta+1, \gamma+1, q, x)}{\phi(\alpha, \beta, \gamma, q, x)}$$

Heine obtained a continued fraction of the form (1) in which

$$a_{2n} = q^{\beta+n-1}\frac{(1-q^{\alpha+n-1})(1-q^{\gamma-\beta+n-1})}{(1-q^{\gamma+2n-2})(1-q^{\gamma+2n-1})},$$

$$a_{2n+1} = q^{\alpha+n-1}\frac{(1-q^{\beta+n})(1-q^{\gamma-\alpha+n})}{(1-q^{\gamma+2n-1})(1-q^{\gamma+2n})}.$$

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* *Math. Annalen*, vol. 33, 1888, p. 258.

† *Crelle*, vol. 34, 1847.

When $n$ increases indefinitely, the limit of both of these coefficients is obviously zero. Theorem 1 is therefore applicable and shows that the continued fraction represents a meromorphic function, which, by the theorem of §9, may be proved to be identical with the above quotient.

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# ON THE DIFFERENTIATION OF AN INFINITE SERIES TERM BY TERM.

By M. B. Porter.

If

$$f(x) = \sum_1^\infty u_n(x), \qquad a < x < b,$$

denote a convergent series of single valued, differentiable, real functions of the real variable $x$, the ordinary sufficient condition that the derivative $f'(x)$ exist and that

$$f'(x) = \sum_1^\infty u_n'(x),$$

namely, that each term $u_n(x)$ be differentiable and that the series $\sum_1^\infty u_n'(x)$ be uniformly convergent, is practically limited to the case where the uniformity can be tested by means of a so called $M$-series. That is, we shall have

$$f'(x) = \sum_1^\infty u_n'(x), \qquad a < x < b,$$

if there exist a convergent series of positive terms, $\sum_1^\infty M_n$, such that

$$|u_n'(x)| \leqq M_n, \qquad a < x < b. \tag{A}$$

The practical importance of this test in that, in particular, it is immediately applicable to power series and thus leads to a simple as well as rigorous treatment of undetermined coefficients, renders an elementary proof of it desirable. The following proof, while probably not new,* has not come to the notice of the writer elsewhere, and seems to be sufficiently elementary.

If $x$ and $x + \Delta x$ lie in the interval of convergence of the series

$$f(x) = \sum_1^\infty u_n(x),$$

* Dini, *Functionen einer reellen Grösse*, p. 154, uses the Mean Value Theorem, but as he deduces a more general criterion, his proof is much less elementary.

then, by the primary definition of convergence, we have

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\sum_{1}^{\infty}\frac{u_n(x+\Delta x)-u_n(x)}{\Delta x},$$

and, by the first Mean Value Theorem,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\sum_{1}^{\infty}u_n'(x+\theta_n\Delta x),\qquad 0<\theta_n<1. \tag{1}$$

If $\epsilon$ be an arbitrarily assigned positive number as small as we please, ($A$) tells us that a positive integer $N$ can be found such that

$$\left|\sum_{N}^{\infty}u_n'(x+\theta_n\Delta x)\right|<\frac{\epsilon}{2}. \tag{2}$$

Furthermore, since each term $u_n(x)$ has a derivative $u_n'(x)$, a number $\delta$ can be found such that, if $|\Delta x|<\delta$,

$$\frac{u_n(x+\Delta x)-u_n(x)}{\Delta x}=u_n'(x+\theta_n\Delta x)$$

will differ from its limit, $u_n'(x)$, numerically by less than $\epsilon/2n$, so that

$$\left|\sum_{1}^{n}u_n'(x+\theta_n\Delta x)-\sum_{1}^{n}u_n'(x)\right|<\frac{\epsilon}{2}. \tag{3}$$

Hence it follows that

$$\left|\frac{f(x+\Delta x)-f(x)}{\Delta x}-\sum_{1}^{\infty}u_n'(x)\right|<\epsilon,\qquad |\Delta x|<\delta,$$

which proves the theorem.

The continuity, or even the integrability, of the derivatives $u_n'(x)$ is not postulated in the above proof, since the Mean Value Theorem merely requires that the derivatives exist.

The same proof can be applied when $x$ denotes the complex variable, $x_1+x_2i$, the Mean Value Theorem* for functions of a complex variable now being employed.

NEW HAVEN, CONNECTICUT,
AUGUST, 1901.

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* Stolz, *Diff. u. Integralrechnung*, vol. 2, p. 95.

# A NOTE ON GEODESIC CIRCLES.

By J. K. Whittemore.

Bianchi defines a geodesic circle as the locus of a point on a surface at a constant geodesic distance from a fixed point of the surface.* In this note I use the term in this sense. Darboux calls a curve of constant geodesic curvature a geodesic circle.† Bianchi states in a footnote‡ that all the geodesic circles of a surface are curves of constant geodesic curvature, when the total curvature of the surface is constant.

In this note I shall prove a set of more general theorems from which Bianchi's statement and its converse follow at once. I use the term "geodesic circle" in Bianchi's sense. My theorems are the following:

Theorem 1. *If, on a surface, there exists a family of concentric geodesic circles such that the geodesic curvature of each curve of the family is constant, then the total curvature of the surface is constant along each curve of the family, and the surface is applicable to a surface of revolution so that the geodesic circles fall on the circles of latitude of this surface.*

Theorem 2. *Conversely, if, on a surface, there exists a family of concentric geodesic circles such that the total curvature of the surface is constant along each curve of the family, then the geodesic curvature of each geodesic circle is constant, and the surface is applicable to a surface of revolution by the former theorem.*

Theorem 3. *Finally, if a surface is applicable to a surface of revolution so that the members of a family of concentric geodesic circles of the surface fall upon the circles of latitude of the surface, then the geodesic circles are curves of constant geodesic curvature and of constant total curvature.*

I suppose the common centre of the geodesic circles to be an ordinary point of the surface. I choose as curvilinear coordinates on the surface, the geodesic distance, $u$, from the common centre of the circles, and the angle, $v$, which a geodesic makes at this point with some fixed direction on the surface at this point. Then the linear element of the surface has the form,

$$ds^2 = du^2 + C^2 dv^2,$$

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* Bianchi. *Vorlesungen über Differentialgeometrie*, p. 160.

† Darboux. *Théorie générale des surfaces*, vol. 3, p. 151.

‡ *l. c.*, p. 162.

where $$C_{\substack{u=0\\ r=r}} = 0, \qquad \left(\frac{\partial C}{\partial u}\right)_{\substack{u=0\\ r=r}} = 1.$$

The geodesic curvature of one of the geodesic circles is given by the formula:

$$\frac{1}{\rho_u} = -\frac{1}{C}\frac{\partial C}{\partial u}.$$

The total curvature of the surface at any point is*

$$\kappa = -\frac{1}{C}\frac{\partial^2 C}{\partial u^2}$$

Consider first Theorem 3. The element of arc $dS$ of a surface of revolution can be expressed in the form

$$dS^2 = d\sigma^2 + \phi(\sigma)^2 d\theta^2,$$

where $d\sigma$ denotes the element of arc of a meridian curve, $y = f(x)$, and $\phi(\sigma) = y$; the longitude being measured by $\theta$. According to the hypothesis of the theorem, it is possible to determine $u$ and $v$ as functions of $\sigma$ and $\theta$ such that, when $u = \text{const.}$, $\sigma = \text{const.}$ also and furthermore that $dS^2 = ds^2$, or

$$du^2 + C^2 dv^2 = d\sigma^2 + \phi^2 d\theta^2. \qquad (1)$$

Let $u = u(\sigma, \theta)$, $v = v(\sigma, \theta)$. Then, from the first condition it follows that

$$\frac{\partial u}{\partial \theta} = 0,$$

Substituting for $du$ and $dv$ their values in terms of $d\sigma$ and $d\theta$, we show further that

$$\frac{\partial u}{\partial \sigma} = 1, \qquad \frac{\partial v}{\partial \sigma} = 0.$$

Hence $u = \sigma$ and $v$ is a function of $\theta$ alone. Since the assignment of the individual geodesics $v = v_1$ to the individual meridians $\theta = \theta_1$ is still arbitrary, we may make this assignment in such a way that $v = \theta$. It then follows from (1) that

$$C^2 = \phi(\sigma)^2,$$

*i. e.*, that $C$ is a function of $u$ alone. The same will be true of $1/\rho_u$ and $\kappa$; hence the theorem.

---

* For these formulas see Bianchi, *l. c.* pp. 161, 148, 159.

Let us now suppose, to prove Theorem 1, that the geodesic circles, $u =$ constant, are curves of constant geodesic curvature. Then

$$\frac{1}{\rho_u} = -\frac{1}{C}\frac{\partial C}{\partial u} = -U'$$

where $U$ is a function of $u$ alone. We have by integration

$$C = e^{U+V}$$

where $V$ is a function of $v$ alone.

Now, since $\left(\frac{\partial C}{\partial u}\right)_{u=0} = 1$, it follows that

$$1 = U'(0)e^{U(0)+V(v)},$$

and hence that $V$ is a constant. $C$ is, therefore, a function of $u$ alone, and Theorem 1 is proved.*

Finally, to prove Theorem 2, suppose that the geodesic circles, $u =$ constant, are for the surface curves of constant total curvature. Then is

$$\kappa = -\frac{1}{C}\frac{\partial^2 C}{\partial u^2} = -F(u) \qquad (2)$$

where $F(0)$ is finite, since the centre of the geodesic circles is an ordinary point of the surface. In order to integrate equation (2), let $U$ be a solution of the ordinary differential equation

$$U'' + U'^2 = F(u)$$

which vanishes for $u = 0$. Then we have

$$\frac{1}{C}\frac{\partial^2 C}{\partial u^2} = U'' + U'^2$$

Next, replace $C$ by $z$ where

$$C = z\,e^U;$$

$z$ is an unknown function of $u$ and $v$, which must however vanish for $u = 0$. For the determination of $z$ we have the equation:

$$\frac{\partial^2 z}{\partial u^2} + 2U'\frac{\partial z}{\partial u} = 0.$$

---

* For a proof of the theorem that, if the element of arc of a surface can be written in the form $dS^2 = Edu^2 + Gdv^2$, where $E$, $G$ are functions of $u$ alone, the surface is applicable to a surface of revolution, *cf.* Picard, *Traité d'analyse*, vol. 1, p. 423.

Integrating this equation, the general value of $z$ is found to be

$$z = e^V \int_0^u e^{-2U}\, du + V_1,$$

where $V$ and $V_1$ are arbitrary functions of $v$. Since now $z$ and $u$ vanish together, $V_1$ is identically zero, and

$$C = e^{U+V} \int_0^u e^{-2U}\, du.$$

Furthermore, since for $u = 0$, we have

$$\frac{\partial C}{\partial u} = 1, \qquad U = 0,$$

it follows that for all values of $v$

$$e^V = 1.$$

Hence $V$ is identically zero, and

$$C = e^u \int_0^u e^{-2U}\, du$$

and is a function of $u$ alone. Theorem 2 follows at once.*

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS, MAY 18, 1901.

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* Picard, *l. c.*

# NOTE ON THE FUNCTIONS DEFINED BY INFINITE SERIES WHOSE TERMS ARE ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE; WITH CORRESPONDING THEOREMS FOR DEFINITE INTEGRALS.

By W. F. Osgood.

Let

$$f_1(x) + f_2(x) + \cdots \qquad (a)$$

be an infinite series of real functions of the real variable $x$, $f_n(x)$ being single valued and continuous throughout the interval $(a, b)$: $a \leqq x \leqq b$, and let the series converge for all values of $x$ pertaining to this interval. Its value shall be denoted by $F(x)$. The class of functions $F(x)$ thus defined and the mode of convergence of the series have been studied at length.* In particular, the function $F(x)$ need not be continuous throughout any interval lying in $(a, b)$. This class is coextensive with the class which, at first sight, might appear to be smaller and which is obtained by imposing on $f_n(x)$ the further restriction that it be *analytic* throughout the interval $(a, b)$. For, let

$$s_n(x) = f_1(x) + \cdots + f_n(x).$$

Then there exists a function $\sigma_n(x)$ analytic throughout the interval $(a, b)$ and differing numerically from $s_n(x)$ uniformly by an arbitrarily small amount, *i. e.* $\sigma_n(x)$ satisfies the relation

$$|s_n(x) - \sigma_n(x)| < \epsilon,$$

where $\epsilon$ is an arbitrarily small positive quantity.† It is readily seen that the function $\sigma_n(x)$ converges towards the value $F(x)$; for

$$|F(x) - s_n(x)| < \epsilon', \qquad \left.\begin{matrix}\epsilon', \text{ arbitrary;}\\ n, \text{ suitably chosen}\end{matrix}\right\};$$

$$|s_n(x) - \sigma_n(x)| < \epsilon;$$

$$\therefore \quad |F(x) - \sigma_n(x)| < \epsilon + \epsilon'.$$

Now form the series:

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* *Cf.* Schoenflies, Bericht über die Mengenlehre, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8, 1900, chap. 7, p. 217.

† $\sigma_n(x)$ may, in fact, be chosen as a polynomial. This follows from Weierstrass's theorem that any continuous function of a real variable can be represented by a uniformly convergent series of polynomials. *Cf.* Picard, *Traité d'analyse*, vol. 1, 1st ed., p. 258.

$$\phi_1(x) + \phi_2(x) + \cdot \cdot \cdot \cdot \cdot,$$

where $\phi_n(x) = \sigma_n(x) - \sigma_{n-1}(x)$, $n > 1$; $\phi_1(x) = \sigma_1(x)$. Then this is the desired series; namely, a series of analytic functions having the same function $F(x)$ for its value as the original series.

If we pass from the domain of real to that of complex variables and functions, the precise analogon would be a series of complex functions, single valued and continuous throughout a two-dimensional region $T$ of the complex $z$-plane and convergent at each point of this region. Such series are without interest unless their terms are analytic functions of $z$. We shall here consider *the class of functions*

$$F(z) = f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot$$

*defined by a series of functions, each of which is analytic throughout a two-dimensional region T of the complex z-plane, the series being assumed to converge for every point z lying within this region.*

The principal results of the paper are stated in Theorems I and II, and are of a very general nature. The condition of uniform convergence is nowhere imposed. In fact, Theorem II presupposes only the bare convergence of the series — nothing more. The proof of Theorem II depends on a theorem in the theory of sets of points, which is quoted in No. 5, and which has already proved of value in the study of general questions in the Theory of Functions.

We observe that from Theorem II follows at once a proof of the theorem that *a function of z which is continuous throughout T cannot be represented by means of a series of functions each analytic throughout T unless it is at least in parts of T analytic.*

## §1.

1. THEOREM I. *If the infinite series*

$$f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot, \tag{1}$$

*whose terms are all functions of z, single valued and analytic throughout a region T of the complex z-plane, converges for all values of z pertaining to a set of points which is everywhere dense throughout T (and which, in particular, may be enumerable); and if, furthermore, the relation*

$$|f_1(z) + \cdot \cdot \cdot + f_n(z)| \leqq G$$

*holds for all points z of this set (and hence of T) and for all values of n, G*

*being a positive constant, then the series converges for all values of $z$ in $T$ and the function $F(z)$ defined by the series:*

$$F(z) = f_1(z) + f_2(z) + \cdots$$

*is analytic throughout $T$.*

To prove the theorem, we set

$$f_n(z) = u_n(x, y) + iv_n(x, y), \qquad z = x + iy,$$

form the series of harmonic functions

$$u_1(x, y) + u_2(x, y) + \cdots, \tag{2}$$

and show that this series converges uniformly throughout the neighborhood of every interior point $A$ of $T$. Hence the function $u(x, y)$ defined by (2) is harmonic in $T$ and the series can be differentiated term by term, the resulting series converging likewise uniformly. The series of conjugate functions,

$$v_1(x, y) + v_2(x, y) + \cdots \tag{3}$$

converges for each interior point of $T$ and the function $v(x, y)$ that it defines is conjugate to the function $u(x, y)$.

2. The details of the proof are as follows. We begin by expressing certain harmonic functions with which we have to deal by means of Poisson's Integral.* About the point $A$ describe a circle lying wholly within $T$. Denote its radius by $R$, any interior point $(x, y)$ by $P$, and any point on the circumference by $Q$. Further, denote the angles that $AP$ and $AQ$ make with the positive axis of $x$ by $\phi$ and $\psi$ respectively, and the distance $AP$ by $r$. Then, if $w(x, y)$ is any function, continuous throughout the region composed of the interior and the circumference of the circle and harmonic at all points within the circle, and if the values of $w(x, y)$ along the circumference of the circle are denoted by $W(\psi)$, Poisson's Integral gives the following representation of $w(x, y)$ at the interior points of the circle:

$$w(x, y) = \frac{1}{2\pi} \int_0^{2\pi} W(\psi) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2} d\psi. \tag{4}$$

Consider the function

$$s_n(x, y) = u_1(x, y) + \cdots + u_n(x, y).$$

* *Cf.*, for example, Picard's *Traité d'analyse*, vol. 2, p. 16.

*Its partial derivatives,* $\frac{\partial s_n}{\partial x}$ *and* $\frac{\partial s_n}{\partial y}$ *, are finite throughout the neighborhood of the point* $A$. We will choose as this neighborhood, $\mathfrak{R}_A$, the interior of a circle with $A$ as centre and radius $\frac{1}{2}R$. From (4),

$$s_n(x, y) = \frac{1}{2\pi}\int_0^{2\pi} S_n \frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\, d\psi;$$

hence

$$\frac{\partial s_n}{\partial x} = \frac{1}{2\pi}\int_0^{2\pi} S_n \frac{\partial}{\partial x}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\, d\psi,$$

with a similar formula for $\partial s_n/\partial y$. Differentiation under the sign of integration is here allowable.* Denote by $K$ a quantity that is at least as large as the maximum numerical value of either of the expressions

$$\frac{\partial}{\partial x}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}, \qquad \frac{\partial}{\partial y}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2},$$

$(x, y)$ being any point of $\mathfrak{R}_A$ and $0 \leqq \psi < 2\pi$. Then, since $|S_n| \leqq G$ for all values of $\psi$, it follows that

$$\left|\frac{\partial s_n}{\partial x}\right| \leqq KG, \qquad \left|\frac{\partial s_n}{\partial y}\right| \leqq KG, \qquad q.\ e.\ d.$$

3. We are now in a position to prove that the function $s_n(x, y)$ converges uniformly toward a limit throughout the region $\mathfrak{R}_A$. The proof follows at once from a theorem which has been stated and proved for functions of a single variable by de la Vallée-Poussin.† The extension to functions of $n$ variables is immediate. The theorem is as follows.

*Let* $s_n(x)$ *be a function of* $x$, *continuous throughout the interval* $a \leqq x \leqq b$ *and having a finite derivative*‡ *for all values of* $x$ *in the interval and for* $n = 1, 2 \ldots$ ; *i. e.*

$$|s_n'(x)| < L$$

---

* *Cf.* Picard, *Traité d'analyse*, vol. 1, p. 29.

† *Annales de la Société scientifique de Bruxelles*, première partie, vol. 17, 1893, p. 8.

‡ Instead of a finite derivative it is enough to require that the difference-quotient remain finite, *i. e.* that

$$|s_n(x + h) - s_n(x)| \leqq Nh,$$

where $x$ and $x + h$ are any two points of the interval, $n = 1, 2, \ldots$, and $N$ is a constant.

*for all such values of $x$ and $n$, $L$ being a constant. Furthermore, let $s_n(x)$ converge toward a limit for each one of a set of values of $x$ everywhere dense throughout this interval, when $n = \infty$. Then $s_n(x)$ converges uniformly toward a limit for the totality of values of $x$ pertaining to the interval.*

Thus the function $s_n(x, y)$ is seen to converge uniformly toward a limit, $u(x, y)$, throughout the region consisting of $\mathfrak{R}_A$ and its boundary. $s_n(x, y)$ is a harmonic function throughout $\mathfrak{R}_A$ and it takes on a continuous set of boundary values which, when $n = \infty$, converge uniformly toward a limit. The limit of $s_n(x, y)$ is, therefore, harmonic throughout $\mathfrak{R}_A$* and

$$\frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \cdots \cdots .$$

This series converges uniformly. The corresponding theorems are true for the partial derivatives of $s_n(x, y)$ and $u(x, y)$ with respect to $y$.

4. It remains to show that the series (3) converges and defines a function conjugate to $u(x, y)$. Consider the series

$$\sum_{n=1}^{\infty} \int_{(x_0, y_0)}^{(x, y)} \frac{\partial u_n}{\partial x}\, dy - \frac{\partial u_n}{\partial y}\, dx. \tag{5}$$

Since the series

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial x}, \qquad \frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial y}$$

converge uniformly, these series can be integrated term by term and the value of the expression (5) is

$$\bar{v}(x, y) = \int_{(x_0, y_0)}^{(x, y)} \frac{\partial u}{\partial x}\, dy - \frac{\partial u}{\partial y}\, dx,$$

which is a function conjugate to $u(x, y)$. On the other hand, the value of the general term of (5) is $v_n(x, y) - v_n(x_0, y_0)$. Hence, the series

$$\sum_{n=1}^{\infty} v_n(x, y) - v_n(x_0, y_0)$$

converges and, if $(x_0, y_0)$ is chosen as one of the points for which the series

* This follows from a theorem of Harnack's. *Cf.* Harnack, *Math. Annalen*, vol. 35, 1889, p. 22, or Picard, *Traité d'analyse*, vol. 2, p. 56. That such a series can be differentiated term by term and that the resulting series is uniformly convergent can readily be shown by means of the representation by Poisson's Integral.

(1) and hence the series $\sum_{n=1}^{\infty} v_n(x_0, y_0)$ converges, it is seen that the series (3) converges. If $v(x, y)$ denotes its value, then

$$v(x, y) = \bar{v}(x, y) + v(x_0, y_0),$$

and the function $v(x, y)$ is conjugate to $u(x, y)$,

This completes the proof of Theorem I.

§2.

5. THEOREM II. *If the infinite series*

$$f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot ,$$

*whose terms are all functions of $z$, single valued and analytic throughout a region $T$ of the complex $z$-plane, converges for all values of $z$ lying within this region, then there exists one or more analytic functions $F_1(z)$, $F_2(z)$, . . . such that the domain of definition of $F_i(z)$ coincides, in part or wholly, with a region $t_i$ lying in $T$ and, for all values of $z$ in $t_i$,*

$$F_i(z) = F(z),$$

*where $F(z)$ denotes the value of the series. Every interior point of $T$ either lies within some $t_i$ or else a part of one or more regions $t_i$ extends into the neighborhood of this point.**

*The functions $F_i(z)$ may be infinite in number. They are always enumerable.*

Let $t$ be an arbitrary, two-dimensional region lying wholly within $T$. It may happen that, for a suitably chosen (large) positive quantity $M$, the absolute value of

$$S_n(z) = f_1(z) + \cdot \cdot \cdot \cdot f_n(z)$$

is less than $M$ for all points $z$ of $t$ and for all values of $n$. In that case, it follows from Theorem I that the series defines a function analytic throughout $t$.

* This is not the same as saying that the point lies on the boundary of one or more regions $t_i$. This case is included in the statement. But it may happen that the point lies on the boundary of no region $t_i$, but is a point of condensation of boundaries of an infinite number of regions $t_i$.

It may be objected that the notation $F_i(x)$, $i = 1, 2, \ldots$, assumes from the start that these functions are enumerable. No use, however, is made of this assumption in the proof, and so it is a matter of no moment. An adequate notation could be devised by allowing $i$ to take on other than integral values, since the number of the functions $F_i(z)$ obviously does not exceed the power (Mächtigkeit) of the continuum.

In any case, I say that *it is always possible so to choose $M$ that, throughout some two-dimensional region $t'$ lying in $t$, the relation holds:*

$$|\mathfrak{S}_n(z)| \leqq M$$

*for all values of $z$ in $t'$ and for $n = 1, 2, \ldots$, and hence $F(z)$ is analytic throughout $t'$,* $F(z)$ denoting the value of the series.

Suppose this statement were not true. Choose a set of positive quantities $M_i$ increasing indefinitely with $i$. Denote the points of $t$ at which $|\mathfrak{S}_n(z)| \leqq M_i$ for all values of $n$ by $P_i$. Then the assumption is that the points of $P_i$ nowhere in $t$ form a two-dimensional continuum. They cannot, then, even be everywhere dense throughout any such continuum, since $|\mathfrak{S}_n(z)|$ is a continuous function of $z$. The points of $P_i$ are all found in $P_{i+1}$. Denote by $P$ the set of points, each of which ultimately appears in some $P_i$:

$$P = \lim_{i=\infty} P_i.$$

Now each point of $t$ ultimately appears in some $P_i$; hence $P$ is identical with the totality of points of $t$. But $t$ is a two-dimensional continuum. We are thus led to a contradiction of a theorem of the theory of sets of points, which says:*

*In a two-dimensional region $t$ of a plane let sets of points $P_1, P_2, \ldots$ be given, which have the following properties:*

1) *the points of $P_i$ are all contained among the points of $P_{i+1}$;*

2) *in no two-dimensional region whatever are the points of $P_i$ everywhere dense.*

*Furthermore, let*

$$P = \lim_{i=\infty} P_i$$

*be the totality of points that participate in the sets $P_i$. Then no part of $P$ can form a two-dimensional continuum.*

6. To complete the proof of the theorem, let $A$ be any interior point of $T$. If $A$ lies in a region $t_i$, the theorem is granted. If not, consider an arbitrary neighborhood $t$ of $A$. Then there exists one or more regions $t_i$ having points in common with $t$. But $t$ is *any* neighborhood of $A$.

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* This theorem (with an unessential restriction) I obtained for the case of one-dimensional regions in an article on Non-Uniform Convergence and the Integration of Series Term by Term, *Am. Jour. of Math.*, vol. 19, 1897, p. 173. The proof there given is indirect. For a simple, direct proof *cf. Math. Annalen*, vol. 53, 1900, p. 462.

Finally, the regions $t_i$ are enumerable. This is a theorem of the theory of sets that has long since been proved. In fact, the internal area* of each $t_i$ is a positive quantity $I_i$, and the sum of any number of these quantities does not exceed the internal area of $T$ itself. If, then, we choose a set of steadily decreasing quantities $\epsilon_1, \epsilon_2, \ldots$ with $\lim_{n=\infty} \epsilon_n = 0$, there will be but a finite number of $t_i$'s for each of which $I_i > \epsilon_1$. These we write in the first row of a table and arrange at pleasure in that row. Then there will be but a finite number of further $t_i$'s for which $I_i > \epsilon_2$. These we write in the second row of the table and arrange at pleasure in that row. And so on. Each $t_i$ must ultimately appear in some row of the table, and this proves the theorem.

7. The question arises as to whether the number of the regions $t_i$ can ever actually be infinite. The answer is in the affirmative, as can readily be shown by means of methods due to Runge.† Thus, for example, a series can be constructed satisfying all the conditions of the theorem for the interior of the square bounded by the axes of reals and pure imaginaries and by the lines $x = 1$ and $y = 1$, whose value $F(z)$ is $\frac{1}{2}$ when $x \geqq \frac{1}{2}$; $\frac{1}{3}$, when $x \geqq \frac{1}{3}$ but $< \frac{1}{2}$; $\frac{1}{4}$, when $x \geqq \frac{1}{4}$ but $< \frac{1}{3}$; etc.

Two or more of the functions $F_i(z)$ may be capable of being deduced from one another by means of analytic continuation, so that they form parts of one and the same analytic function. This, again, can be demonstrated by Runge's methods. Thus one can, for example, construct a series convergent for the square just considered and having the value $\frac{1}{2}$ when $x \neq \frac{1}{2}$, the value 1 when $x = \frac{1}{2}$. In fact, we may draw any enumerable set of lines whatever in $T$, each of which has its extremities in two distinct points of the boundary of $T$, the lines not cutting themselves or each other or clustering about any point lying within $T$.‡ Then we may assume arbitrary values along these lines, provided merely that the values for a given line are such as an analytic function could take on along that line. To each of the regions $t_i$ into which $T$ is thus divided we may assign an arbitrary function $F_i(z)$ analytic throughout this region. Then we can construct a series whose value $F(z)$ shall correspond to the values and the functions thus assumed.

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* For the definition of the *internal area* of a set of points *cf.*, for example, Jordan, *Cours d'analyse*, vol. 1, 2d ed., 1893, §36.

† Runge, *Acta Math.*, vol. 6, 1885, p. 229.

‡ Examples in which these conditions are not fulfilled can also be readily constructed.

§3

8. By means of Theorem I, a theorem of Stieltjes may be generalized. Stieltjes proved in substance the following theorem :*

*If*

$$f_1(z) + f_2(z) + \cdots$$

*is a series of functions, each single valued and analytic throughout a region $T$ of the $z$-plane, and if*

$$|f_1(z) + \cdots + f_n(z)| < G$$

*for all values $z$ of $T$ and for all values of $n$, $G$ being a constant; if, furthermore, the above series converges uniformly throughout some two-dimensional region $t$ lying in $T$, then the series converges throughout the entire region $T$ and its value is a function $F(z)$ which is analytic throughout $T$.*

From Theorem I it follows that the requirement of *uniform convergence throughout $t$* can be replaced by the less restrictive one that the series *converge for each point of a set everywhere dense throughout $t$.* Such a set may always be so assumed as to be enumerable.

§4.

9. Theorems I and II were stated for infinite series. Corresponding theorems may be stated for definite integrals and the proofs will hold without modification.

Let $\phi(a, z)$ be a (complex) function of the real variable $a$ in the interval $\mathsf{A}$: $a_0 \leqq a \leqq a_1$ and of the complex variable $z$ in a region $T$ of the $z$-plane. For each value of $a$ in $\mathsf{A}$, $\phi(a, z)$ shall be an analytic function of $z$ throughout $T$; and for each value of $z$ in $T$, $\phi(a, z)$ shall be a continuous function of $a$ throughout $\mathsf{A}$.† Then the integral

$$\int_{a_0}^{a_1} \phi(a, z)\,da$$

will converge for each value of $z$ in $T$. Since its value is invariant of the mode of division of the interval $\mathsf{A}$, we may choose a special mode depending on a parameter $n$ that passes through only positive integral values; for

* Stieltjes, *Annales de la Faculté des Sciences de Toulouse*, vol. 8, 1894, p. J. 56.

† This condition may be replaced by the broader one that $\phi(a, z)$ be an integrable function of $a$ throughout $\mathsf{A}$.

example, we may divide the interval into $n$ equal parts. Let this be done and set

$$s_n(z) = \sum_{i=1}^{n} \phi\,(a_0 + i\,\Delta a,\ z)\,\Delta a, \qquad \Delta a = \frac{a_1 - a_0}{n}.$$

Then $s_n(z)$ is a function of $z$ analytic throughout the region $T$ for each positive integral value of $n$. If it satisfies the further conditions of Theorem I or II respectively, then the same conclusions will hold regarding the function

$$F(z) = \int_{a_0}^{a_1} \phi(a, z)\,da$$

as in the former case for the function $F(z)$ defined by the series.

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# NIM, A GAME WITH A COMPLETE MATHEMATICAL THEORY.

By Charles L. Bouton.

The game here discussed has interested the writer on account of its seeming complexity, and its extremely simple and complete mathematical theory.* The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it.

**1. Description of the Game.** The game is played by two players, $A$ and $B$. Upon a table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no two piles shall be equal at the beginning. A play is made as follows:—The player selects one of the piles, and from it takes as many counters as he chooses; one, two, . . ., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins.

It is the writer's purpose to prove that if one of the players, say $A$, can leave one of a certain set of numbers upon the table, and after that plays without mistake, the other player, $B$, *cannot* win. Such a set of numbers will be called a *safe combination*. In outline the proof consists in showing that if $A$ leaves a safe combination on the table, $B$ at his next move cannot leave a safe combination, and whatever $B$ may draw, $A$ at his next move can again leave a safe combination. The piles are then reduced, $A$ always leaving a safe combination, and $B$ never doing so, and $A$ must eventually take the last counter (or counters).

**2. Its Theory.** A *safe combination* is determined as follows: Write the number of the counters in each pile in the binary scale of notation,† and

* The modification of the game given in §6 was described to the writer by Mr. Paul E. More in October, 1899. Mr. More at the same time gave a method of play which, although expressed in a different form, is really the same as that used here, but he could give no proof of his rule.

† For example, the number 9, written in this notation, will appear as

$$1\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0 = 1001.$$

place these numbers in three horizontal lines so that the units are in the same vertical column. If then the sum of *each* column is 2 or 0 (*i. e.* congruent to 0, mod. 2), the set of numbers forms a safe combination. For example,

1 0 0 1,
1 0 1,
1 1 0 0,

or 9, 5, 12 is a safe combination. It is seen at once that if any two numbers be given, a third is always uniquely determined which forms a safe combination with the two given numbers. Moreover, it is obvious that if $a$, $b$, $c$ form a safe combination any two of the numbers determine the remaining one, that is, the system is closed. A particular safe combination which is used later is that in which two piles are equal and the third is zero. In the proofs which follow, the binary scale of notation is used throughout.

THEOREM I. *If A leaves a safe combination on the table, B cannot leave a safe combination on the table at his next move.* $B$ can change only *one* pile, and he must change one. Since when the numbers in two of the piles are given the third is uniquely determined, and since $A$ left the number so determined in the third pile (*i. e.*, the pile from which $B$ draws) $B$ cannot leave that number. Hence $B$ cannot leave a safe combination.

THEOREM II. *If A leaves a safe combination on the table, and B diminishes one of the piles, A can always diminish one of the two remaining piles, and leave a safe combination.* Consider first an example. Suppose $A$ leaves the safe combination *nine*, *five*, *twelve*, and that $B$ draws *two* from the first pile, leaving the numbers *seven*, *five*, *twelve*, or

1 1 1,
1 0 1,
1 1 0 0.

If $A$ is to leave a safe combination by *diminishing* one of the piles, it is clear that he must select the third pile, that containing *twelve*. The number which is safe with 111 and 101 is 10, or *two*. Hence $A$ must leave *two* in the pile which contains *twelve*, or draw *ten* from that pile, and by doing so he leaves a safe combination.

To prove the general theorem, let the numbers, expressed in the binary scale, be written with the units in a vertical column, and suppose that $A$ left a safe combination. $B$ selects one of the piles and diminishes it. When a number of the binary scale is diminished it is essential to notice that in going

over the number from left to right the first change which occurs is that some 1 is changed to 0, for if a 0 were changed to 1 the number would be increased whatever changes were made in the subsequent digits.* Consider, then, this first column, counting from the left, in which a change occurs. One and only one of the other two numbers will contain 1 in this same column, for $A$ left a safe combination. Let $A$ select the pile which contains the 1 in this column, and change the number by writing 0 in this column, and filling the remaining columns to the right with 0 or 1 so as to make a safe combination. The columns to the left remain unchanged, since they already have the required form. The new number so formed will be *less* than that in the pile which $A$ selected. Hence whatever $B$ draws, $A$ can always diminish one of the piles, and leave a safe combination. That is, if $A$ at any play can leave a safe combination on the table, he can do so at every subsequent play, and $B$ never can do so.

If the play continues in this way $A$ must win. For one of the piles must be reduced to zero by either $A$ or $B$. If $B$ reduces it to zero, the two remaining piles will be unequal, since $B$ can never leave a safe combination, and $A$ at his next move will make them equal, and will thereafter always leave them equal. $B$ must, therefore, reduce the second pile to zero, and $A$ then takes all of the third pile, and wins. If, on the other hand, $A$ is the first player to reduce one of the piles to zero, he leaves the other two piles equal and wins as before. Hence we see that the player who can first leave a safe combination on the table should win.

If it happens that in the beginning a safe combination is placed on the table, the second player should win. If in the beginning a safe combination is not placed on the table, it is easily seen that the first player can always leave a safe combination by diminishing some one of the piles, and he can often do this by drawing from either one of the three piles. Therefore in this case the first player should win. That is, the first player should win or lose according as a safe combination is not or is placed on the table at the beginning.

**3. The Chance of a Safe Combination.** Assuming that the number in each pile at the beginning was determined by chance, let us compute the chance of a safe combination's being placed upon the table. It is easily shown that if each pile contains less than $2^n$ counters and if no pile is zero (*i. e.* if there are three piles), the possible number of different piles is

* The proof of this statement depends on the fact that the number 100 . . . ($n$ ciphers), or $2^n$, is greater than the number 11 . . . ($n$ ones), or $2^{n-1} + 2^{n-2} + \ldots + 2 + 1 = 2^n - 1$.

$\frac{2^n - 1}{2 - 1}$

$$\frac{2^{n-1}(2^{2n}-1)}{3}.$$

The number of safe combinations in the same case is

$$\frac{(2^{n-1}-1)(2^n-1)}{3}.$$

Hence the chance of a safe combination's being placed upon the table at first is

$$\frac{2^{n-1}-1}{2^{n-1}(2^n+1)},$$

and this is the chance that the second player should win. The chances of the first player's winning are to those of the second as

$$2^n+2+\frac{3}{2^{n-1}-1} \quad \text{to} \quad 1,$$

on the assumption that both players know the theory, and that the numbers in the various piles were determined by chance.

**4. A List of Safe Combinations,** $n=4$. The following are the 35 safe combinations all of whose piles are less than 16:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 2 | 4 | 6 | 3 | 4 | 7 | 4 | 8 | 12 |
| 1 | 4 | 5 | 2 | 5 | 7 | 3 | 5 | 6 | 4 | 9 | 13 |
| 1 | 6 | 7 | 2 | 8 | 10 | 3 | 8 | 11 | 4 | 10 | 14 |
| 1 | 8 | 9 | 2 | 9 | 11 | 3 | 9 | 10 | 4 | 11 | 15 |
| 1 | 10 | 11 | 2 | 12 | 14 | 3 | 12 | 15 | | | |
| 1 | 12 | 13 | 2 | 13 | 15 | 3 | 13 | 14 | | | |
| 1 | 14 | 15 | | | | | | | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 8 | 13 | 6 | 8 | 14 | 7 | 8 | 15 |
| 5 | 9 | 12 | 6 | 9 | 15 | 7 | 9 | 14 |
| 5 | 10 | 15 | 6 | 10 | 12 | 7 | 10 | 13 |
| 5 | 11 | 14 | 6 | 11 | 13 | 7 | 11 | 12 |

Of course, to give *all* safe combinations of numbers less than 16 we should have to add to the above table the 15 of the form $0, n, n$.

**5. Generalization.** The foregoing game can be at once generalized to the case of any number of piles, with the same rule for playing. In this case a safe combination is a set of numbers such that, when written in the binary scale and arranged with the units in the same vertical column, the sum of each column is even (*i. e.*, $\equiv 0$, mod. 2). Just as before, it is shown that the

player who first leaves a safe combination can do so at every subsequent play, and will win. The induction proof is so direct that it seems unnecessary to give it.

**6. Modification.** The game may be modified by agreeing that the player who takes the last counter from the table *loses*. This modification of the three pile game seems to be more widely known than that first described, but its theory is not quite so simple.

A safe combination is defined just as in the first case, *except* that 1, 1, 0 is *not* a safe combination, and 1, 1, 1 and 1, 0, 0 *are* safe combinations. When the first theory indicates that $A$ should play 1, 1, 0 he must play either 1, 1, 1 or 1, 0, 0. The earlier part of the proof proceeds as before. In order to complete it, we must show that $B$ can never leave 1, 1, 1; that, when 1, 1, 0 is indicated for $A$, he can always play either 1, 0, 0 or 1, 1, 1; and finally that, if the play is carried out in this way, $B$ must take the last counter. That $B$ can never leave 1, 1, 1 is at once clear, for $A$ never leaves 1, 1, $n$ where $n > 1$, since this is not a safe combination. Secondly, let us consider what sets of numbers $B$ can leave which would indicate 1, 1, 0 as $A$'s next play in the first form of game. They are 1, 1, $n$ where $n > 1$, and 1, $n$, 0 where $n > 1$. In the first case $A$ leaves 1, 1, 1 and in the second 1, 0, 0. The proof is now easily completed. Either $A$ or $B$ reduces a pile to zero. If $B$ does so, the other two piles are unequal and both greater than unity, or at least one of the two remaining piles is unity. In the latter case $A$ obviously wins. In the former case $A$ makes the two piles equal, and then keeps them equal until $B$ reduces one of them to 1 or 0. If $B$ makes it 1, $A$ takes all the other pile; if $B$ makes it 0, $A$ takes all but 1 of the other pile. Hence if $B$ first reduces a pile to zero $A$ wins. If $A$ first reduces a pile to zero he leaves the other two piles equal and each greater than unity, and wins as before. Hence if $A$ plays on the safe combinations as here modified, $B$ must take the last counter from the table, and *loses*. That is, in this modified game, also, the player who can first get a safe combination should win.

This modified game can also be generalized to any number of piles. The safe combinations are the same as before, *except* that an odd number of piles, each containing one, is now safe, while an even number of ones is not safe.

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# ON THE GROUPS GENERATED BY TWO OPERATORS OF ORDER THREE WHOSE PRODUCT IS ALSO OF ORDER THREE.

By G. A. Miller.

Two operators of order two always generate a dihedral rotation group whose order is twice the order of their product.* In this case, the group generated by two operators is completely defined by the orders of the operators and that of their product. Similarly, the group generated by two operators of order three, whose product is of order two, is completely defined by the orders of the two generating operators and that of their product. That is, the alternating group of order 12 is the only one which satisfies these conditions.† On the contrary, two operators $(s_1, s_2)$ of order three whose product is also of this order may generate any one of an infinite system of groups of a finite order. The present paper is devoted to the determination of some of the properties of the groups of this system.‡

The following conjugate commutators

$$s_3 = s_1^2 s_2^2 s_1 s_2, \qquad s_4 = s_2^2 s_1^2 s_2^2 s_1 s_2^2 = s_1 s_2 s_1^2 s_2^2, \qquad s_5 = s_2 s_1^2 s_2^2 s_1$$

of such a group $(G)$§ are commutative since $s_3 s_4 s_5 = s_4 s_3 s_5 = 1$. The last equation may readily be verified by means of the equations $(s_1 s_2)^3 = s_1 s_2 s_1 s_2 s_1 s_2 = 1$ or $s_1 s_2 s_1 = s_2^2 s_1^2 s_2$. Hence $s_3, s_4, s_5$ generate an abelian subgroup $(H)$ of $G$ which is either cyclic or may be generated by two independent operators. We shall soon see that $H$ is the commutator subgroup of $G$. From the equations

$$s_1^2 s_3 s_1 = s_1 s_2^2 s_1 s_2 s_1 = s_4, \qquad s_1^2 s_4 s_1 = s_2 s_1^2 s_2^2 s_1 = s_5, \qquad s_1^2 s_5 s_1 = s_3,$$

it follows that $H$ is invariant in $G$. Its order $(h)$ is the order $(g)$ of $G$ divided by either 3 or 9, since the group generated by $s_2$ and $H$ contains $s_1^2 s_2^2 s_1 = s_2^2 s_3$ and hence is also invariant in $G$. This proves that $H$ is the

* *Bull. Amer. Math. Soc.*, vol. 7, 1901, p. 424.

† *Cf.* Burnside, *Theory of Groups of Finite Order*, 1897, p. 291.

‡ *Cf. ibid.*, p. 296.

§ In what follows the symbol $G$ will always represent a group of the system under consideration.

commutator subgroup of $G$.* Moreover, the given equations show that $s_1$ and $s_2$ transform the operators $s_3, s_4, s_5$ in the same manner. Having determined some of the properties which all of the groups under consideration possess, we proceed to determine some of the special properties which depend upon the factors of $h$ and especially to determine the necessary and sufficient conditions that such groups exist.

When $H$ is cyclic and of order $p^a$, $p$ being any prime number, the matter is quite simple. If $p \equiv 2 \bmod 3$, it follows from the above equations that $G$ is abelian.† In this case $s_3 s_4 s_5 = s_3^3 = 1$ and therefore $h = 1$. The non-cyclic group of order 9 is clearly the only group which satisfies these conditions. It is the only abelian group in the infinite system of groups under consideration and will not be considered in what follows. If $p \equiv 1 \bmod 3$, $s_2$ and $H$ generate one and only one group of this system for every value of $a > 0$. Its order is $3h$ and it contains $2h$ operators of order 3. The direct product of any one of these groups and an operator of order 3 is of order $9h$ and belongs to this system.

That there is only one group of order $9h$ for every value of $a$ may be proved as follows: Such a $G$ is isomorphic to the group of order 3 with respect to those of its operators which are commutative with each operator of $H$. That is, $G$ contains a cyclic invariant subgroup of order $3h$ and hence an invariant subgroup of order 3. It also contains an invariant subgroup of order $3h$, generated by $H$ and $s_2$, which does not include any invariant group of order 3. In other words, it is the direct product mentioned above. Hence *there are two and only two groups for every value of* $a > 0$ *whenever* $p \equiv 1 \bmod 3$. When $p = 3$, it follows from $s_3 s_4 s_5 = 1$ that $H$ is either the identity or of order 3.‡ In this case there is therefore only one group; viz. the non-abelian group of order 27 which includes no operator of order 9.

From what precedes, it follows that we can construct at least one group of the required system whenever $H$ is cyclic and all the prime factors of $h$ are $\equiv 1 \bmod 3$, or when $h$ is three times such a number. Moreover, it is not difficult to see that the equation $s_3 s_4 s_5 = 1$ cannot be satisfied under any other conditions, since $H$ is the direct product of its cyclic subgroups, each of whose orders is a power of a single prime. It remains to consider the cases when

* *Quarterly Jour. of Math.*, vol. 28, 1896, p. 266; *Cf.* Weber, *Algebra*, vol. 2, 1899, p. 131.

† The group of isomorphisms of the cyclic group of order $p^a$ is of order $p^{a-1}(p-1)$.

‡ Burnside, *l. c.*, p. 76.

$H$ is non-cyclic, but is generated by two of its operators. We shall again begin with the case when $h = p^a$.

When $p \equiv 2 \bmod 3$, $s_3$ and $s_4$ must be the independent generators of $H$, since the cube of a common operator is identity. Similarly, we observe that $s_3$ and $s_4$ cannot have more than 3 common operators whenever $p = 3$. When $p \equiv 1 \bmod 3$, we may construct a $G$ by establishing a $(p^\alpha, p^\beta)$ isomorphism between two $G$'s of orders $3p^\alpha$, $3p^\beta$ respectively in such a manner that the resulting group contains subgroups of order $p$ which are not invariant. Hence, in this case, the two invariants of $H$ may have any orders such that their product is equal to $h$; *i. e.* $s_3$ and $s_4$ may have a common subgroup of any order less than the order of $s_3$.

It has been observed that the two invariants of $H$ are equal when $p \equiv 2 \bmod 3$. It may therefore be assumed that the order of $s_3$ and $s_4$ is $p^m$, where $m = \frac{a}{2}$, and that $H$ is of type $(m, m)$. $H$ contains $p^{m-1}(p+1)$ cyclic subgroups of order $p^m$, which may be divided into $p+1$ sets such that each one of those of one set has only identity in common with any one of another set. These subgroups are transformed by the holomorph of $H$ according to a transitive substitution group of degree $p^{m-1}(p+1)$ and of order $p^{3m-2}(p^2-1)$. Since $p \equiv 2 \bmod 3$, the group of isomorphisms of $H$ contains operators of order 3 which permute each of its cyclic subgroups of order $p^m$.

Such an operator $(s_1)$ transforms the $p+1$ sets mentioned above as units. It cannot transform any one of these sets into itself, since such a set is composed of $p^{m-1}$ groups of order $p^m$. Hence $H$ and $s_1$ generate a group which contains $2h$ operators of order 3. For $s_2$ we may select any substitution which transforms the operators of $H$ in the same manner as $s_1$ does and which gives a product of order $p^m$ when multiplied by $s_1$. This proves the existence of the groups in question for every value of $p \equiv 2 \bmod 3$ and for every value of $m > 0$.

When $p = 3$ and the two independent generators $(t_1, t_2)$ of $H$ are either of order $3^m$ or of orders $3^m$ and $3^{m-1}$ respectively, it is easy to verify that there is an operator of order 3 in the group of isomorphisms of $H$, which transforms $t_1$ into $t_2 t_1$ and $t_2$ into $t_1^{-3} t_2^{-2}$. This operator, $s_1$, and $H$ generate a group of order $3h$ which contains $2h$ operators of order 3 that are not found in $H$; for, $(t_1^\alpha t_2^\beta s_1)^3 = t_1^\alpha t_2^\beta t_2^{-\alpha} t_1^{-2\alpha} t_2^\beta t_1^{3\beta} t_2^\alpha t_1^\alpha t_1^{-3\beta} t_2^{-2\beta} = 1$. If we take for $s_2$ an operator such that $s_1^2 s_2 = t_1$, it is evident that $s_1$, $s_2$ generate $G$. This proves the existence of groups whenever $p = 3$ and $H$ has two invariants of order $3^m$ or one invariant of order $3^m$ and the other of order $3^{m-1}$. It was proved above

that these conditions are also necessary. It remains, therefore, only to examine the case when $H$ has two independent generators whose orders are not powers of the same prime number.

In this general case, we may write $h$ in the form $3^{a_0}p_1^{a_1}p_2^{a_2}\ldots$ and observe that each of the subgroups whose orders are $3^{a_0}, p_1^{a_1}, p_2^{a_2}, \ldots$ is either cyclic or the direct product of two cyclic groups.* The necessary and sufficient conditions that $s_1$ and $s_2$ can be so chosen as to generate a group which contains any one of these subgroups, as $H$, have been determined. If these conditions are satisfied for each of the given subgroups, we may suppose $s_1$ and $s_2$ so formed as to generate the group formed by establishing a $(3^{a_0}, p_1^{a_1}, p_2^{a_2}\ldots)$ isomorphism between the separate group whose $h$'s are powers of primes. If they are not satisfied in each instance, it follows from the equation $s_3s_4s_5 = 1$ that there is no group in the infinite system under consideration which has this $H$. Hence this general case is included under the special cases considered above.

CORNELL UNIVERSITY, JULY, 1901.

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## ON THE INVARIANTS OF A QUADRANGLE UNDER THE LARGEST SUBGROUP, HAVING A FIXED POINT, OF THE GENERAL PROJECTIVE GROUP IN THE PLANE.**

BY W. A. GRANVILLE.

IN the ANNALS OF MATHEMATICS, vol. 12, p. 82, Professor Lovett proposes the problem of finding the invariants of a quadrangle under the transformations of the six parameter group in the plane generated by the infinitesimal transformations:

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q},$$

$$p = \frac{\partial f}{dx}, \qquad q = \frac{\partial f}{dy}.$$

This is the largest subgroup of the general projective group in the plane, which has a fixed point. The invariants found by Professor Lovett were

* *Cf. Bull. Amer. Math. Soc.*, vol. 7, 1901, p. 424.

** Presented to the American Mathematical Society at its meeting, April 27, 1901.

$$c_1 = \frac{y_3}{y_4} \cdot \frac{x_4y_2 - x_2y_4}{x_3y_2 - x_2y_3}, \qquad c_2 = \frac{x_2}{x_3} \cdot \frac{x_1y_3 - x_3y_1}{x_1y_2 - x_2y_1}.$$

On trial it will be found that these functions do not satisfy the differential equations of the complete system (6) on page 82:

$$\begin{gathered}\sum_{i=1}^{4} x_i \frac{\partial \phi}{\partial x_i} = \sum_{i=1}^{4} y_i \frac{\partial \phi}{\partial x_i} = \sum_{i=1}^{4} x_i \frac{\partial \phi}{\partial y_i} = \sum_{i=1}^{4} y_i \frac{\partial \phi}{\partial y_i} = 0, \\ \sum_{i=1}^{4} \left\{ x_i^2 \frac{\partial \phi}{\partial x_i} + x_i y_i \frac{\partial \phi}{\partial y_i} \right\} = \sum_{i=1}^{4} \left\{ x_i y_i \frac{\partial \phi}{\partial x_i} + y_i^2 \frac{\partial \phi}{\partial y_i} \right\} = 0; \end{gathered} \tag{6}$$

and hence cannot be the invariant functions.

The two solutions of (6) are found to be

$$c_1 = \frac{\left(\frac{y_1}{x_1} - \frac{y_3}{x_3}\right)\left(\frac{y_2}{x_2} - \frac{y_4}{x_4}\right)}{\left(\frac{y_1}{x_1} - \frac{y_4}{x_4}\right)\left(\frac{y_2}{x_2} - \frac{y_3}{x_3}\right)} = \frac{m_{01} - m_{03}}{m_{01} - m_{04}} \cdot \frac{m_{02} - m_{04}}{m_{02} - m_{03}};$$

and

$$c_2 = \frac{\left(\frac{y_1}{x_1} - \frac{y_1 - y_3}{x_1 - x_3}\right)\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_2}{x_1 - x_2}\right)}{\left(\frac{y_1}{x_1} - \frac{y_1 - y_4}{x_1 - x_4}\right)\left(\frac{y_1 - y_3}{x_1 - x_3} - \frac{y_1 - y_2}{x_1 - x_2}\right)} = \frac{m_{10} - m_{13}}{m_{10} - m_{14}} \cdot \frac{m_{14} - m_{12}}{m_{13} - m_{12}};$$

where $m_{ik}$ is the slope of the line drawn from the point $(x_i, y_i)$ to the point $(x_k, y_k)$, the origin being denoted by $(x_0, y_0)$. Hence the theorem:

*If a quadrangle* (1234) *be transformed by the Lie group*

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q},$$

*the cross ratio of the pencil of lines drawn from any vertex of the pentagon* (01234) *to the remaining four vertices remains constant.*

If the problem be considered from the standpoint of projective geometry, this result follows at once from the well known fact that the cross ratio of a pencil of any four concurrent lines is an invariant under any projective transformation. It is also evident that two of the cross ratios here considered are independent.

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ANNALS OF MATHEMATICS.

Published in October, January, April, and July, under the auspices of Harvard University, Cambridge, Mass., U. S. A.

*Cambridge:* Address *The Annals of Mathematics*, 2 University Hall, Cambridge, Mass., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75c. All drafts and money orders should be made payable to Harvard University.

*London:* Longmans, Green & Co., 39 Paternoster Row. Price, 2 shillings a number.

*Leipzig:* Otto Harrassowitz, Querstrasse 14. Price, 2 marks a number.

Printed by The Salem Press Co., Salem, Mass., U. S. A.